

# Circular Waves in Thermoelastic Plates Bordered with Viscous Liquid

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**Abstract-** The paper concentrates on the study of propagation of thermoelastic waves in a homogeneous, transversely isotropic, thermally conducting elastic plate bordered with layers (or half-spaces) of viscous liquid on both sides in the context of non classical theories of thermoelasticity. Complex secular equations for symmetric and antisymmetric wave motion of the circular plate, in completely separate terms, are derived. Finally, in order to illustrate the analytical results, the numerical solution is carried out for transversely isotropic plate of cobalt material bordered with water by using the functional iteration method.

**Index Terms-** Crested waves, viscous, thermal relaxation, Biot's constant, transversely isotropic.

## I. INTRODUCTION

The use of sandwich structures in various applications, especially in many engineering phenomena, including the response of soils, geological materials and composites, the assumption of isotropic behaviour may not capture some significant features of the continuum response. The formulation and solution of anisotropic problems is far more difficult and cumbersome than its isotropic counterpart. There are reasonable grounds for assuming anisotropy in the continents. Anisotropy in the earth crust's and upper mantle affects the wave characteristics considerably. Also, Ultrasonic Lamb waves have been investigated extensively for damage detection in advanced composite materials. Banerjee and Pao [3] investigated the propagation of plane harmonic waves in infinitely extended anisotropic solids taking into account the thermal relaxation time.

Dhaliwal and Sherief [4] extended the generalized thermoelasticity theories to anisotropic elastic bodies. Zhu and Wu [5] brought out detailed analysis of the characteristics of Lamb waves in elastokinetics. Sharma and Singh [6], Sharma and Pathania [7] and Sharma and Walia [8] investigated the propagation of Lamb waves in thermoelastic solid plates bordered by liquid media maintained at uniform temperature. Recently, Sharma and Sharma [9] investigated the propagation of Lamb waves in thermoviscoelastic plate loaded with viscous fluid layer of varying temperature. Hussain, Ahmad and Asawa [10] investigated analytically and through experimentation the discharge capacity of a sharp-crested circular orifice in an open channel under free flow conditions. The present investigation is concerned with the study of circular crested wave propagation in an infinite homogeneous, transversely isotropic, thermoelastic plate

bordered with an viscous liquid layers or half spaces on both sides. It is noticed that the motion for circular crested waves is also governed by Rayleigh-Lamb type secular equations as in the case of a rectangular plate. More general dispersion equations of Lamb type waves are derived and discussed in coupled and uncoupled theory of thermoelasticity. Numerical solution of the dispersion equations for cobalt material has been carried out and presented graphically.

## II. FORMULATION OF THE PROBLEM

We consider an infinite homogeneous, transversely isotropic, thermally conducting elastic plate of thickness  $2d$  initially at temperature  $T_0$ , having a viscous liquid layer of thickness  $h$  on both sides. We take origin of the co-ordinate system  $(r, \theta, z)$  on the middle surface of the plate and  $z$ -axis normal to it along the thickness. The plate is axi-symmetric with respect to  $z$ -axis as axis of symmetry. We take  $r-z$  plane as the plane of incidence. From Sharma and Pathania [7] and Chandrasekhar [1], we have taken the basic governing equations in non-dimensional form for generalized thermoelasticity in the absence of body forces and heat sources and for viscous liquid (homogeneous liquid) medium respectively.

$$u_{,rr} + \frac{1}{r}u_{,r} - \frac{1}{r^2}u + c_2u_{,z} + c_3w_{,z} - (T + t_1\delta_{2k}\dot{T})_{,r} = \ddot{u} \quad (1)$$

$$c_3\left(u_{,z} + \frac{1}{r}u_z\right) + c_2w_{,rr} + \frac{1}{w_z} + c_1w_{,z} - \bar{\beta}(T + t_1\delta_{2k}\dot{T})_{,z} = \ddot{w} \quad (2)$$

$$T_{,rr} + \frac{1}{r}T_{,r} + KT_{,z} - (\dot{T} + t_2\dot{T}) = \delta\left(\frac{\partial}{\partial t} + t_3\delta_{12}\frac{\partial^2}{\partial t^2}\right)\left(u_{,r} + \frac{1}{r}u + \beta_{1z}v_z\right) \quad (3)$$

$$v_z\frac{\partial}{\partial t}\nabla^2\bar{u}_{Lj} + \left(\delta_z^2 + \frac{1}{3}v_z\frac{\partial}{\partial t}\right)\nabla\nabla\cdot\bar{u}_{Lj} - \frac{\bar{\beta}_z}{\bar{\rho}}\nabla T_{Lj} = \ddot{\bar{u}}_{Lj}, \quad (4)$$

$$T_{Lj} = -\frac{\varepsilon_z\rho_z\delta_z^2}{\beta_z\rho}\nabla\cdot\bar{u}_{Lj}, \quad (j=1,2)$$

where

$$\varepsilon = \frac{\beta_z^2 T_0}{\rho C_e c_{11}}, \beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3, \beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3,$$

$$\frac{\bar{\beta}_z}{\bar{\rho}} = \frac{\beta_z}{\rho}, \delta_z^2 = \frac{c_z^2}{v_z^2}, c_z^2 = \frac{\lambda_z}{\rho_z}, \beta_z^* = 3\lambda_z\alpha^*, \varepsilon_z = \frac{\beta_z^* T_0}{\rho_z C_e^* \lambda_z} \quad (5)$$

Here  $t_0$  and  $t_1$  are the thermal relaxation times.  $\delta_{ik}$  is the Kronecker's with  $k = 1$  for Lord-Shulman (LS) theory and  $k = 2$  for Green-Lindsay (GL) theory of generalized thermoelasticity.  $\varepsilon$  is the thermomechanical coupling constant and  $v_1$  is the longitudinal wave velocity in the thermoelastic half-space.  $\vec{u}(r, z, t) = (u, 0, w)$  is the displacement vector,  $T(r, z, t)$  is the temperature change,  $T_0$  is the reference temperature of solid,  $c_L$  is the velocity of sound in the liquid,  $\alpha^*$  is the coefficient of volume thermal expansion.  $\varepsilon_L$  is the thermomechanical coupling in the liquid. Along the  $z$  direction for the top liquid layer ( $j = 1$ ) and for the bottom liquid layer ( $j = 2$ ).  $c_L$ ,  $\rho_L$  are velocity of sound, the density of the liquid respectively,  $\lambda_L$  is the bulk modulus,  $C_V^*$  is the specific heat at constant volume of fluid.  $T_{L_j}$  is the temperature deviation of liquid medium from ambient temperature  $T_0^*$ .

$\vec{u}_{L_j}(r, z, t) = (u_{L_j}, 0, w_{L_j})$  ( $j = 1, 2$ ) is the velocity vector.  $u_{L_j}$  and  $w_{L_j}$  are respectively. Along the direction for the top liquid layer and for the bottom liquid layer. Moreover liquid is assumed to be thermally non-conducting hypothetical solid. Here the dot notation is used for time differentiation and comma denotes spatial derivatives. In the liquid layers, in addition to introducing non-dimensional quantities, we take

$$u_{L_j} = \frac{\partial \phi_{L_j}}{\partial r} + \frac{\partial \psi_{L_j}}{\partial z}, \quad w_{L_j} = \frac{\partial \phi_{L_j}}{\partial z} - \frac{\partial \psi_{L_j}}{\partial r} - \frac{\psi_{L_j}}{r} \quad (6)$$

where  $\phi_{L_j}$  and  $\psi_{L_j}$ , ( $j = 1, 2$ ) are respectively, the scalar and vector velocity potential function, we obtain,

$$\nabla^2 \phi_{L_j} - \frac{1}{\left( \delta_L^2 (1 + \varepsilon_L) + \frac{4}{3} v_L \frac{\partial}{\partial t} \right)} \ddot{\phi}_{L_j} = 0 \quad (7)$$

$$\nabla^2 \psi_{L_j} - \frac{1}{v_L} \dot{\psi}_{L_j} = 0 \quad (8)$$

$$T_{L_j} = - \frac{\varepsilon_L \rho_L \delta_L^2}{\beta_L \rho} \nabla^2 \phi_{L_j}, \quad (j = 1, 2) \quad (9)$$

### III. BOUNDARY CONDITIONS

Boundary conditions at the solid-liquid interfaces  $z = \pm d$  are:

$$\begin{aligned} \sigma_{zz} &= (\sigma_{zz})_L, \quad \sigma_{rz} = (\sigma_{rz})_L, \quad u = u_L, \quad w = w_L, \\ T_{L_j} + H(T - T_{L_j}) &= 0 \end{aligned} \quad (10)$$

where  $H$  is Biot's heat transfer coefficient

### IV. SOLUTION OF THE PROBLEM

We assume solutions of the form

$$\{u, w, T\} = \{J_0(\xi r), VJ_0(\xi r), WJ_0(\xi r)\} S e^{i\omega t - \alpha z} \quad (11.1)$$

$$\{\phi_{L_j}, \psi_{L_j}\} = \{\Phi_{L_j} J_0(\xi r), \Psi_{L_j} J_0(\xi r)\} S e^{i\omega t - \alpha z} \quad (11.2)$$

where  $c = \frac{\omega}{\xi}$  is the non-dimensional phase velocity,  $\omega$  is

the frequency and  $\xi$  is the wave number.  $J_0, J_1$  are respectively, the Bessel functions of order zero and one.  $\theta$  is the angle of inclination of wave normal with axis of symmetry.  $m$  is still unknown parameter.  $V, W$  respectively are, the amplitude ratios of displacement  $w$ , temperature  $T$  and  $\Phi_{L_j}, \Psi_{L_j}$  are the amplitude ratios of velocity potentials  $\phi_{L_j}, \psi_{L_j}$  respectively.

Upon using solution (11.1) in (1) - (3). We obtain

$$m^6 + Am^4 + Bm^2 + C = 0 \quad (12)$$

Here

$$\begin{aligned} A &= \frac{P - Jc^2}{c_1 c_2} + \frac{1 - c^2 \tau_0}{K} + \frac{i \xi^2 c^3 \bar{\beta}^2 \tau_0' \tau_1}{c_1 K} \\ B &= \frac{(c_2 - c^2)(1 - c^2)}{c_1 c_2} + \frac{(1 - c^2 \tau_0)(P - Jc^2)}{c_1 c_2 K} + \frac{i \xi^2 c^3 \tau_0' \tau_1}{c_1 c_2 K} [c_1 - 2c_3 \bar{\beta} + (1 - c^2) \bar{\beta}^2] \\ C &= \frac{c_2 - c^2}{c_1 c_2 K} [(1 - c^2 \tau_0)(1 - c^2) + i \xi^2 c^3 \tau_0' \tau_1] \end{aligned}$$

$$P = c_1 + c_2 - c_3, J = c_1 + c_2, \tau_0 = t_0 + i\omega^{-1}, \tau_0' = t_0' + i\omega^{-1}, \tau_1 = t_1 + i\omega^{-1} \quad (13)$$

Equation (12) being cubic in  $m^2$  admits six solutions for

$m$ , with property  $m_{2n} = -m_{2n-1}$ ,  $n = 1, 2, 3$ . On seeking the solution of (1)-(3) in the form (11.1), displacements, temperature change can be expressed as

$$\begin{aligned} u &= \sum_{q=1}^3 (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_0(\xi r) e^{-i\omega t} \\ w &= \sum_{q=1}^3 V_q (-A_q \sin \xi m_q z + B_q \cos \xi m_q z) J_0(\xi r) e^{-i\omega t} \\ T &= \sum_{q=1}^3 W_q (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_0(\xi r) e^{-i\omega t} \end{aligned} \quad (14)$$

where  $m_q$  are the roots of (12).  $A_q, B_q$  are amplitudes.  $V_q, W_q$  are amplitude ratios which can be expressed as

$$V_q = \begin{cases} -im_q, & q=1, 3 \\ \frac{i\bar{\beta}}{m_q} & q=5 \end{cases}, \quad W_q = \begin{cases} \frac{(c_2 + c_3 m_q^2 + 1 - c^2)}{c \xi}, & q=1, 3 \\ \frac{[c c_2 m_q^2 + (P - Jc^2) m_q^2 + (c_2 - c^2)(1 - c^2)]}{c \bar{\beta} [c_2 m_q^2 + 1 - c_3(\bar{\beta} - c^2)]}, & q=5 \end{cases} \quad (15)$$

where

$$a_q = \bar{\beta} \frac{c_2 m_q^2 + (1 - c_3 / \bar{\beta}) - c^2}{(c_1 - c_3 \bar{\beta}) m_q^2 + c_2 - c^2}, \quad b_q = (c_2 m_q^2 + 1 - c^2 + i c \tau_1 \bar{\beta} T_q) \frac{1}{c_3}$$

On using stress strain relation

$$\sigma_{zz} = (c_3 - c_2) \left( u_z + \frac{1}{r} \right) + c_1 w_z + i \omega \tau_1 \bar{\beta} T_z, \quad \sigma_{rz} = u_z + w_z,$$

axial stress and shear stress in  $r-z$  plane is obtained as

$$\sigma_{zz} = -\xi \sum_{q=1}^5 P_q (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_0(\xi r) e^{-i \omega t}$$

$$\sigma_{rz} = i \xi c_2 \sum_{q=1}^5 f_q (-A_q \sin \xi m_q z + B_q \cos \xi m_q z) J_1(\xi r) e^{-i \omega t} \quad (16)$$

where

$$P_q = \begin{cases} c_2(1+a_q) - c^2 a_q, & q=1, 3 \\ \frac{c_2 m_q^2 - (c_2 - c^2) b_q}{m_q^2}, & q=5 \end{cases}, \quad f_q = \begin{cases} (1+a_q) m_q, & q=1, 3 \\ \frac{m_q^2 - b_q}{m_q}, & q=5 \end{cases} \quad (17)$$

On seeking solution of form (14.2) for (4) in liquid layers  $d < z < d+h$  and  $-(d+h) < z < -d$ , such that the acoustical pressure vanishes at  $z = \pm(d+h)$ , we obtain

$$(\Phi_{z_1}, T_{z_1}) = (1, S_z \xi^2) A_7 \sin \xi m_7 [z - (d+h)] e^{-i \omega t} \quad (18)$$

$$(\Phi_{z_2}, T_{z_2}) = -(1, S_z \xi^2) A_8 \sin \xi m_8 [z + (d+h)] e^{-i \omega t} \quad (19)$$

$$\Psi_{z_1} = A_9 \sin \xi m_9 [z - (d+h)] e^{-i \omega t} \quad (20)$$

$$\Psi_{z_2} = -A_{10} \sin \xi m_{10} [z + (d+h)] e^{-i \omega t} \quad (21)$$

where  $A_q, q = 7, 8, 9, 10$  are amplitudes.

$$S_z = \frac{\varepsilon_z \rho_z \delta_z^2 c^2}{\bar{\beta}_z \rho_z [\delta_z^2 (1 + \varepsilon_z) - \frac{4}{3} i \omega v_z]}, \quad m_{7,8} = \pm \sqrt{c^2 \left( \delta_z^2 (1 + \varepsilon_z) - \frac{4}{3} i \omega v_z \right) - 1},$$

$$m_{9,10} = \pm \sqrt{\frac{i c^2}{v_z \omega} - 1} \quad (22)$$

## V. DERIVATION OF THE SECULAR EQUATIONS

By invoking the interface conditions (13), we obtain a system of ten simultaneous linear equations. System of equations have a non-trivial solution if the determinant of the coefficients of  $A_q$  vanishes, which leads to a characteristic equation for the propagation of modified guided thermoelastic waves in the plate. The characteristic equation for the thermoelastic plate waves in this case, after applying lengthy algebraic reductions of the determinant leads to the following secular equations

$$\left[ \frac{T_1}{T_2} \right]^{+1} - \frac{P_{35} G_2}{P_{19} G_1} \left[ \frac{T_1}{T_2} \right]^{+1} + \frac{P_{35} G_7}{P_{19} G_1} \frac{T_7}{[T_2]^{+1}} + \frac{\bar{F}_7}{P_{19} G_1} \frac{T_7}{[T_2]^{+1}} = -\frac{P_{35} G_2}{P_{19} G_1} \quad (23)$$

where

$$P_7 = -\bar{\rho} v_z^{-1} i \xi (i \xi c^2 - 4 v_z \xi^2 + c \xi^2 m_7^2), \quad P_9 = -2 \bar{\rho} i \xi^2 c m_9,$$

$$f_7 = -2 i \bar{\rho} \xi^2 c m_7, \quad f_9 = -\bar{\rho} i \xi^2 c (m_9^2 - 1), \quad V_7 = \xi m_7 = U_9$$

$$V_9 = -\xi = U_7, \quad W_7 = S_z \xi^2 \bar{F}_9 = f_{19} G_1' - f_{35} G_7' + f_{35} G_7' + f_{79} G_7',$$

$$P_{q9} = (P_q U_9 - U_q P_9), \quad f_{q9} = (f_q V_9 - V_q f_9), \quad q = 1, 3, 5, 7 \quad (24.1)$$

$$G_1 = \begin{vmatrix} f_1 & f_3 & -f_5 \\ V_1 & V_3 & -V_5 \\ W_1(m_1 - HT_1) & W_3(m_3 - HT_3) & 2W_5 HT_5 \end{vmatrix},$$

$$G_1' = \begin{vmatrix} P_1 T_1^{+1} & P_3 T_3^{+1} & -P_5 T_5 \\ U_1 T_1^{+1} & U_3 T_3^{+1} & -U_5 T_5 \\ W_1(m_1 - HT_1) & W_3(m_3 - HT_3) & 2W_5 HT_5 \end{vmatrix} \quad (24.2)$$

Here

$$T_k = \tan m_k, \quad k = 1, 3, 5, \quad T_k = \tan \gamma^* m_k, \quad k = 7, 9, \quad \gamma^* = \xi d,$$

$G_3, G_5, G_7$  and  $G_3', G_5', G_7'$  can be written from  $G_1$  and  $G_1'$  by replacing subscript  $(3, 5, 7)$  of elements in  $G_1$  and  $G_1'$  with  $(1, 5, 7), (1, 3, 7), (1, 3, 5)$  respectively with the exception that  $(1, 3), (2, 3)$  element in both  $G_7$  and  $G_7'$  are with positive sign and  $(3, 3)$  element is given by  $W_5(m_5 - HT_5)$ . The superscript +1 corresponds to asymmetric and -1 refers to symmetric modes.

The Rayleigh-Lamb type equation also governs circular crested thermoelastic waves in a plate bordered with layers of viscous liquid or half space viscous liquid on both sides. Although the frequency wave number relationship holds whether the waves are straight or circularly crested, the displacement and stress vary according to Bessel functions rather than trigonometric functions as far as the radial coordinate is concerned. For large values of  $r$ , we have

$$J_0(\xi r) \rightarrow \frac{\sin \xi r + \cos \xi r}{\sqrt{\pi \xi r}}, \quad J_1(\xi r) \rightarrow \frac{\sin \xi r - \cos \xi r}{\sqrt{\pi \xi r}} \quad (25)$$

Thus, far from the origin the motion becomes periodic in  $r$ . As becomes very large, the straight crested behavior is the limit of circular crested waves.

The secular equation (28) is the transcendental equation and contains complete information about the phase velocity, wave number and attenuation coefficient of the plate waves. In general, wave number and hence the phase velocity of the waves is complex quantity, therefore the waves are attenuated in space.

In order to find real phase velocity and attenuation coefficient, we write

$$c^{-1} = V^{-1} + i \omega^{-1} Q \quad (26)$$

so that  $\xi = R + i Q$ , where  $V = \frac{\omega}{R}$ ,  $R$  and  $Q$  are real.

Here  $V$  is the propagation speed and  $Q$  the attenuation coefficient of the waves. Upon using relation (26) in (23) the values of  $V$  and  $Q$  for different modes have been obtained numerically.

## VI. SPECIFIC LOSS

The ratio of energy dissipated  $\Delta W$  in taking a specimen through a cycle, to the elastic strain energy ( $W$ ) stored in the specimen when strain is maximum is called specific loss. It is the most direct method of defining the internal friction for material. According to Kaliski [2] for a sinusoidal plane

wave of small amplitude, the specific loss  $\left| \frac{\Delta W}{W} \right|$  equals

$4\pi$  times the absolute value of imaginary part of  $\xi$  to the real part of  $\xi$ , i.e.

$$\text{Specific loss} = \left| \frac{\Delta W}{W} \right| = 4\pi \frac{\text{Im}(\xi)}{\text{Re}(\xi)} = 4\pi \left| \frac{Q}{R} \right| = \left| \frac{VQ}{\omega} \right|$$

## VII. UNCOUPLED THERMOELASTICITY

In case of uncoupled thermoelasticity (UCT), the thermomechanical-coupling constant vanishes, i.e.  $\varepsilon = 0$ , hence (15) reduces to

$$m^4 + \bar{B}m^2 + \bar{C} = 0 \text{ and } m^2 = \frac{\xi^2(1 - c^2\tau_0)}{\bar{K}} \quad (27)$$

$$\text{where } \bar{B} = -\frac{\xi^2(P - Jc^2)}{c_1c_2}, \bar{C} = \frac{\xi^4(c_2 - c^2)(1 - c^2)}{c_1c_2}$$

Consequently the secular equation (23) in this case reduces to

$$\left[ \frac{T_1}{T_2} \right]^{\pm} + \frac{f_1 P_1 F_1 T_1}{f_2 P_2 T_2} \left[ \frac{T_1}{T_2} \right]^{\pm} + \frac{f_1 T_1 P_1 T_1}{f_2 P_2 T_2} \left[ \frac{T_1}{T_2} \right]^{\pm} - \frac{f_1 T_1 P_1 T_1}{f_2 P_2 T_2} \left[ \frac{T_1}{T_2} \right]^{\pm} - \frac{f_1 T_1 P_1 T_1}{f_2 P_2 T_2} \left[ \frac{T_1}{T_2} \right]^{\pm} = \frac{f_1 P_1 T_1}{f_2 P_2 T_2} \quad (28)$$

where  $P_q, f_q, q=1,5,7, P_q, f_q, q=5,7, P_q, f_q, q=5$  are given by (41) and (31.1).

$$V_1 = i \frac{c_2 m_1^2 + 1 - c^2}{c_3 m_1}, V_5 = i \frac{c_3 m_5}{c_1 m_5^2 + c_2 - c^2}, W_1 = 0 = W_5 \quad (29)$$

$m_q, q=1,5$  are the roots of (28)

For inviscid liquid ( $\mu_L = 0$ ), at uniform temperature we get the secular equation same as obtained and discussed by Sharma and Pathania [6].

## VIII. NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the theoretical results obtained in the previous section, some numerical results are presented. The materials and liquid chosen for this purpose is zinc and water. From Sharma and Pathania [7] and Sharma and Sharma [9], we have taken physical data.

$$\begin{aligned} \rho &= 8.836 \times 10^3 \text{ Kg m}^{-3}, c_{11} = 3.071 \times 10^{11} \text{ Nm}^{-2}, c_{12} = 1.650 \times 10^{11} \text{ Nm}^{-2} \\ c_{13} &= 1.027 \times 10^{11} \text{ Nm}^{-2}, c_{33} = 3.581 \times 10^{11} \text{ Nm}^{-2}, c_{44} = 1.510 \times 10^{11} \text{ Nm}^{-2}, \\ \beta_1 &= 7.04 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}, \beta_3 = 6.90 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}, \varepsilon = 1.29 \times 10^{-2} \\ K_1 &= 0.690 \times 10^9 \text{ W m}^{-1} \text{ deg}^{-1}, K_3 = 0.690 \times 10^9 \text{ W m}^{-1} \text{ deg}^{-1}, \omega = 1.88 \times 10^4 \text{ s}^{-1} \\ C_e &= 4.27 \times 10^2 \text{ JK gm}^{-1} \text{ deg}^{-1}, d = 1, h = 0.25, H = 0.1 \\ c_L &= 1.5 \times 10^3 \text{ m/s}, \rho_L = 1000 \text{ Kg m}^{-3}, T_0 = 298 \text{ K} \end{aligned}$$

The variations of phase velocity of both symmetric and asymmetric modes of wave propagation for inviscid (ideal) liquid ( $\rho_L \neq 0.0, \mu_L = 0.0$ ), low viscous liquid ( $\rho_L \neq 0.0, \mu_L = 0.1$ ) and high viscous liquid ( $\rho_L \neq 0.0, \mu_L = 1.0$ ) are plotted in Fig. 1 with respect to the wave number for angle of inclination  $45^\circ$ .

It is observed that the phase velocity of lowest (acoustic) asymmetric mode ( $A_0$ ) is observed to increase from zero value at vanishing wave number to become closer to Rayleigh wave velocity at higher wave numbers in direction of wave propagation.

The velocity of the acoustic symmetric mode ( $S_0$ ) become dispersion less i.e. remains constant with variation in wave number. The phase velocity of higher (optical) modes of wave propagation, symmetric and asymmetric; attain quite large values at vanishing wave number, which sharply slashes down to become asymptotically closer to the shear wave speed. The velocity of higher modes is observed to develop at a rate, which is approximately  $n$ -time, the magnitude of the velocity of first mode ( $n=1$ ).

The surface wave propagates with attenuation due to the radiation of energy into the medium. This radiated energy will be reflected back to the center of the plate by the lower and upper surfaces. In fact, the multiple reflections between the upper and lower surfaces of the plate form caustics at one of the free surface and a strong stress concentration arises due to which wave field becomes unbounded in the limit  $d \rightarrow \infty$ . The unbounded displacement field is characterized by the singularities of circular tangent functions. The variation of attenuation coefficient with wave number for symmetric and asymmetric acoustic modes is represented in Fig. 2.

The attenuation coefficient for acoustic mode has almost negligible variation with wave number. For higher mode of wave propagation, the magnitude of attenuation coefficient increases monotonically to attain a maximum value and then slashes down sharply to zero with the increase in wave number. For first mode of wave propagation for high viscous the magnitude of attenuation coefficient shoots up then slashes down sharply to zero with the increase in wave number.

Fig. 3 shows the variations of specific loss with wave number for symmetric and asymmetric modes for angle of inclination. The variations of specific loss factor of energy dissipation of symmetric and asymmetric acoustic modes are plotted with respect to wave number.

The trend and behavior of symmetric and asymmetric profiles is noticed to be dispersive. The profile of asymmetric mode attains local maxima at  $R = 1, 4$  and minima near about  $R = 3$  before it becomes smooth and stable for  $R > 7$ . Fig. 4 presents variations of specific loss factor of energy dissipation with respect to liquid loading temperature.



The specific loss factor profile of asymmetric acoustic mode remains non-dispersive and varies linearly with liquid loading temperature; the specific loss factor of asymmetric mode for ( $n=0$ ) has small magnitude as compare to the symmetric mode. Further for other asymmetric mode the magnitude is greater than the symmetric mode. The asymmetric mode for ( $n=1$ ), the magnitude is greater than ( $n=2$ ). The profiles of specific loss factor of symmetric and asymmetric modes follow linear and non-dispersive trends at all values of the loading temperature. The magnitude of specific loss factor of asymmetric mode is noticed to be quite high as compared to that for all other considered cases.

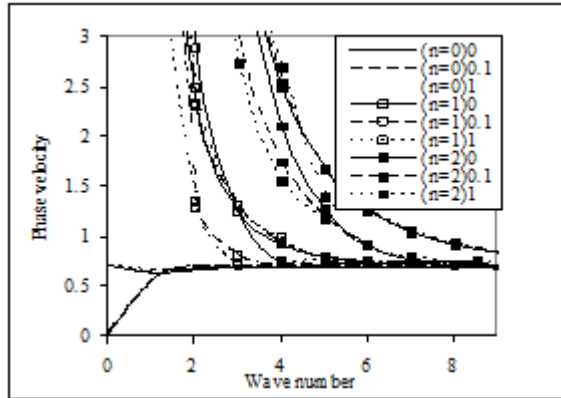


Figure 1. Variation of phase velocity with wave number

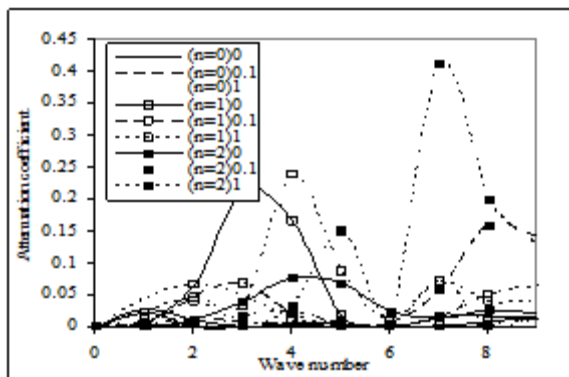


Figure 2. Variation of attenuation coefficient with wave number

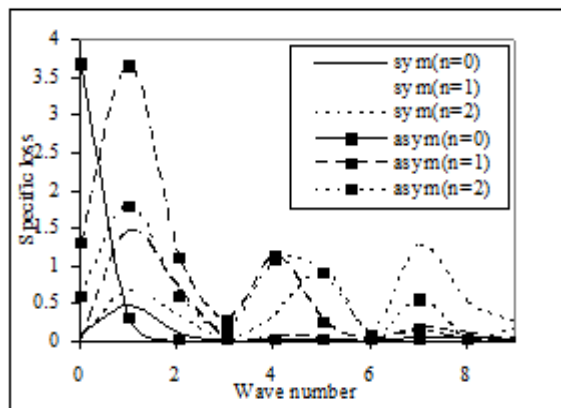


Figure 3. Variation of specific loss with wave number

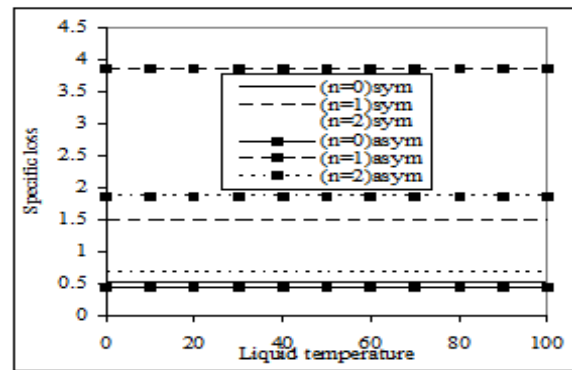


Figure 4. Variation of specific loss with liquid temperature

## CONCLUSIONS

The effect of the viscous loading on the phase velocity, attenuation coefficient due to viscosity have been computed and discussed. It is observed that the inviscid liquid loading does not contribute much to system. However, increasing viscosity of liquid loading magnifies the value of phase velocity of both symmetric and asymmetric modes. The profiles of attenuation coefficient and specific loss factor of acoustic modes are noticed to be highly dispersive. Significant effect of liquid temperature has been observed on specific loss factor of energy dissipation profiles in the considered material plate.

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